

# IMPROVED A PRIORI BOUNDS FOR THERMAL FLUID EQUATIONS

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**ABSTRACT.** We consider two hydrodynamic model problems with three dimensional fluid flow on the torus and temperature-dependent viscosity and conductivity. The ambient heat for the fluid is transported by the flow and fed by the local energy dissipation, modeling the transfer of kinetic energy into thermal energy through fluid friction. Both the viscosity and conductivity grow with the local temperature. We prove a strong a priori bound on the enstrophy of the velocity weighed against the temperature for initial data of arbitrary size, requiring only that the conductivity be proportionately larger than the viscosity. The resulting bounds and conserved quantities are sufficient to establish the existence of global smooth solutions through classical methods.

## 1. INTRODUCTION AND MAIN RESULTS

We first consider the following coupled system of nonlinear PDE arising from the Navier-Stokes Equations with temperature-dependent viscosity and conductivity

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \operatorname{div}(\theta \nabla u) = 0, \quad \nabla \cdot u = 0 \quad (1.1)$$

$$\partial_t(\theta^2) + u \cdot \nabla(\theta^2) - \kappa \operatorname{div}(\theta \nabla(\theta^2)) = \nu \theta |\nabla u|^2 \quad (1.2)$$

on the three-dimensional torus  $\mathbb{T}^3$  with given (sufficiently regular) initial data  $u_0$  and  $\theta_0$ . We require that  $\theta_0(x) \geq 1$  for all  $x \in \mathbb{T}^3$  and that  $u_0$  be mean-zero; (1.1) then forces any smooth solution  $u$  to remain mean-zero for all time. The constant parameters  $\nu$  and  $\kappa$  respectively indicate the strength of the kinematic viscosity and heat conductivity of the fluid. We define

$$K = \kappa/\nu + 1$$

to be (a shift of) the inverse Prandtl number; this parameter will appear many times in the calculations that follow. We remark here that both  $\nu$  and  $\kappa$  should have units of  $K^{-1/2}L^{1/2}T^{-1}$  (here  $K$  is absolute temperature,  $L$  is units of length, and  $T$  is units of time). Modulo the common factor of  $\theta$  (which has units of  $K^{1/2}L^{3/2}$  that must be cancelled),  $\nu$  and  $\kappa$  behave like typical “renormalized dissipations” (with units of  $L^2T^{-1}$ ). Their ratio is therefore unitless (as is the Prandtl number in general), and so is  $K$ . This is important to note since the main results will include conditions on the size of  $K$  and also use  $K$  as an exponent.

Here we use  $\theta^2$  to denote the local temperature of the fluid. However, it will be convenient from now on to work with  $\theta$ , the positive square root of the temperature (see Proposition 2.1); (1.2) then reduces to

$$\partial_t \theta + u \cdot \nabla \theta - \kappa \theta \Delta \theta - 2\kappa |\nabla \theta|^2 = \frac{\nu}{2} |\nabla u|^2 \quad (1.3)$$

Note that the pressure term  $p$  appearing in (1.1) has a different explicit representation than for the standard Navier-Stokes Equations. Specifically,

$$p = (-\Delta)^{-1} \partial_i \partial_j (u_i u_j - \nu \theta \partial_i u_j) = (-\Delta)^{-1} \partial_i (u_j \partial_j u_i - \nu \partial_j \theta \partial_i u_j). \quad (1.4)$$

The existence of solutions for models of fluid equations taking temperature and density into account is a delicate problem, addressed in particular in [8] and [9]. To the best of our knowledge, most investigations of such fluid equations show existence of weak (and admissible) solutions or essentially show that the temperature dependence does not create exotic pathologies (compared with the usual Navier-Stokes equations); see for instance [3]. Since the dissipation has a local dependence on the temperature, many standard techniques for analyzing fluid equations (e.g., a vorticity formulation or methods using paraproduct decomposition) do not readily apply in these cases. Indeed, it can take great effort even to prove that the temperature is bounded

below (see [2] and [15]). Many models (including [2] and [15]) assume from the beginning that the viscosity and conductivity are, as functions of the temperature, bounded above and below (see (1.8) of [11]). They require this to establish a minimal coercivity for the dissipation.

In our models, however, both of these parameters are directly proportional to the square root of the temperature. This is motivated largely by formula (15.3) of [13], the empirical rate at which viscosity increases with temperature in gases. The enhanced Brownian motion causes molecules from one portion of a gas to spread faster throughout the medium, so that the averaged velocities (macroscopic flows) are seen to equalize faster in hotter areas. Observe that, as functions of  $\theta$ , neither the viscosity nor the conductivity are assumed to be bounded above or below; we only assume that  $\theta_0 \geq 1$ . For us, this turns out to be an advantage. One could imagine that thermal viscosity like in (1.1) could produce an advanced regularizing effect for the equation. In regions where the fluid becomes highly turbulent, the local enstrophy generates more heat via (1.2). This, in turn, generates extra viscosity in the trouble-spots. We show that this does indeed happen, and in a way that can be captured by energy-type estimates weighed against the temperature. In these models, it is the *possibility* that the viscosity can blow-up with temperature that prevents turbulent blow-up of the fluid velocity.

We remark here that, for most liquids, the viscosity tends to decrease with temperature [13]. Although the enhanced Brownian motion effect is still present, molecules in liquids experience van der Waals forces or hydrogen bonds, and the added heat disrupts their chemical attraction. Equations (1.1) and (1.2) could then be said to model an incompressible, chemically inert fluid (powder or atmospheric gases). It may be possible to extend our results to more general profiles  $\nu(\theta)$  and  $\kappa(\theta)$ , so long as they both go to  $\infty$  as  $\theta \rightarrow \infty$  (modeling conventional liquids with an unconventional “local evaporation” at high temperatures). This will be the subject of future work.

Our main result establishes a strong a priori bound on smooth solutions to the above system. We prove this in Section 2.

**Theorem 1.1** (Bound for Thermal Navier-Stokes Eq.). *Let  $u$  and  $\theta$  be classical solutions to the system (1.1) and (1.2) on  $\mathbb{T}^3 \times [0, T]$  with initial data  $u_0 \in H^1(\mathbb{T}^3)$  and  $\theta_0 \in L^2(\mathbb{T}^3)$  with  $\theta_0 \geq 1$ . There exists a constant  $L > 0$  independent of all parameters such that if*

$$K \geq \max(L, 2), \quad (1.5)$$

*then there exists a constant  $C$  depending only on  $L$  such that*

$$\int \theta(x, t)^{-1/K} |\nabla u(x, t)|^2 dx \leq \int \theta_0(x)^{-1/K} |\nabla u_0(x)|^2 dx + Ct \frac{K^8}{\nu^{11}} \left( \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \right)^5 \quad (1.6)$$

*holds for all  $t \in [0, T]$ . Moreover, the quantities*

$$\int \theta^{(K-1)/K} |\nabla^2 u|^2 dx, \quad \int \theta^{-(K+1)/K} |\nabla u|^4 dx, \quad \int \theta^{-(K+1)/K} |\nabla u|^2 |\nabla \theta|^2 dx \quad (1.7)$$

*are time integrable on  $[0, T]$ .*

**Remark 1:** The constant  $L$  emerges largely from bounds on Riesz operators and their commutators. We also mention here that  $L$  in fact depends on  $\inf \theta_0$ . Specifically,  $L = L_0(\inf \theta_0)^{-\lambda}$  for some positive  $\lambda$  and  $L_0$  that are truly independent of all parameters. Theorem 1.1 applies equally well to any initial  $u_0$  divergence-free and  $\theta_0$  strictly positive. Theorem 1.1 could likely also be extended to include cases where  $\theta_0 \geq 0$ . Since the temperature essentially satisfies an advected heat equation with nonnegative forcing, one would expect that  $\theta^2$  becomes strictly positive instantaneously. For this article though, we insist that  $\theta_0 \geq 1$  to simplify the argument.

We stress here that the size of the initial data  $u_0$  and  $\theta_0$  (measured, for instance, in any Sobolev norm) can be *arbitrarily large* and that the viscosity  $\nu$  can be *arbitrarily small*. Condition (1.5) also allows the conductivity to be small, so long as the ratio  $\kappa/\nu$  does not go below a fixed constant.

Also, Theorem 1.1 holds for arbitrarily large conductivity. This is intuitively unexpected, since the effect of “generating extra viscosity in trouble-spots” is weaker if the fluid can disperse the ambient temperature more effectively throughout itself; turbulent areas are cooler when  $\kappa$  is larger, and this is seen in the fact that (1.6) degenerates as  $K \rightarrow \infty$ . But the bound still holds, and the thermal weight even improves with larger  $K$  (see Proposition 2.1). This seems to indicate that the build-up of heat in turbulent regions is not indispensable to the boundedness of the solution. However, taking  $\kappa$  to  $\infty$  in the thermal Navier-Stokes system formally leads to the system

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p - \nu \Theta \Delta u &= 0, & \nabla \cdot u &= 0 \\ \Theta(t) &= \sqrt{\Theta_0^2 + \frac{1}{2}\|u_0\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^2}^2} \end{aligned}$$

where the effective viscosity is now time-dependent, but constant in space. Methods of the type employed in this article seem unlikely to prove similar bounds for the above system. See, for instance, the comments of [19].

To further showcase the benefits of thermal viscosity, we also examine a compressible coupled system of nonlinear fluid-like PDE modeled after the Burgers equation. The usual multi-dimensional viscous Burgers equation takes the form

$$\partial_t u + u \cdot \nabla u - \nu \Delta u = 0.$$

Although it lacks compressibility and is vector-valued, the above equation has global smooth solutions (see for example [20]) because it satisfies an  $L^\infty$ -maximum principle: taking a dot product of the viscous Burgers equation with  $u$  leads to a drift-diffusion equation for the scalar  $|u|^2$ , which immediately implies a control on  $\|u\|_{L^\infty}$  for all time. This is sufficient to bootstrap to higher regularity [4].

The naïve thermal Burgers equation satisfies a similar maximum principle. As such, we prefer to work on a model for which the base equation is not known to be globally well-posed. We therefore introduce the “reduced” Burgers equation:

$$\partial_t u + u \cdot \nabla u + \frac{1}{\gamma}(\nabla \cdot u)u - \nu \Delta u = 0,$$

for  $2 \leq \gamma < \infty$ . To the best of our knowledge, the above equation has not been shown to be globally well-posed for smooth initial data on  $\mathbb{T}^3$  (or, for that matter, considered at all). It no longer satisfies an a priori  $L^\infty$  bound since we cannot guarantee any fixed sign for the divergence at the maximum of  $|u|^2$ . However, integrating the equation against  $|u|^{\gamma-2}u$  shows that it retains a uniform bound on  $\|u\|_{L^\gamma}$  (plus time-integrability of some other positive quantities coming from the dissipation); the Burgers equation has been reduced from  $L^\infty$  to  $L^\gamma$ . In the special case  $\gamma = 2$ , the a priori estimates are no better than for the Navier-Stokes equations.

As such, in addition to (1.1) and (1.2), we also examine the initial value problem

$$\partial_t u_j + u_i \partial_i u_j + \frac{1}{\gamma} u_j \partial_i u_i - \nu \partial_i (\theta \partial_i u_j) = 0 \tag{1.8}$$

$$\partial_t (\theta^2) + \partial_i (u_i \theta^2) - \kappa \partial_i (\theta \partial_i (\theta^2)) = \nu \theta |\nabla u|^2 \tag{1.9}$$

on the torus  $\mathbb{T}^3$ , with given (sufficiently regular) initial data  $u_0$  and  $\theta_0$ , and a parameter  $\gamma$  as above. We require that  $\theta_0(x) \geq 1$  for all  $x \in \mathbb{T}^3$ . Here  $\theta^2$  represents the temperature of the system (advected by  $u$ , conducted within the medium, and fed by the local dissipation of kinetic energy).

Our second main result establishes a similar strong a priori bound for the exotic thermal system (1.8) and (1.9). We prove this in Section 3.

**Theorem 1.2** (Bound for Thermal Reduced Burgers Eq.). *Assume  $\gamma = 2$  or  $9/4 \leq \gamma \leq 6$ , and that  $\kappa \geq \nu$ . Let  $u$  and  $\theta$  be classical solutions to the system (1.8) and (3.4) on  $\mathbb{T}^3 \times [0, T]$  with initial data  $u_0 \in H^1(\mathbb{T}^3)$  and  $\theta_0 \in L^2(\mathbb{T}^3)$  with  $\theta_0 \geq 1$ . Then there exist constants  $C = C(\nu, \gamma, K, \|u_0\|_{L^\gamma}, \|\theta_0\|_{L^2}) > 0$  and  $M = M(K, \gamma) > 0$  such that*

$$\int \theta(x, t)^{-1/K} |\nabla u(x, t)|^2 dx \leq \int \theta_0(x)^{-1/K} |\nabla u_0(x)|^2 dx + C(t^M + 1), \quad (1.10)$$

holds for all  $t \in [0, T]$ . Moreover, the quantities

$$\int \theta^{(K-1)/K} |\nabla^2 u|^2 dx, \quad \int \theta^{-(K+1)/K} |\nabla u|^4 dx, \quad \int \theta^{-(K+1)/K} |\nabla u|^2 |\nabla \theta|^2 dx \quad (1.11)$$

are time integrable on  $[0, T]$ . In the special case  $\gamma = 2$ , we explicitly have

$$\int \theta(x, t)^{-1/K} |\nabla u(x, t)|^2 dx \leq \int \theta_0(x)^{-1/K} |\nabla u_0(x)|^2 dx + \bar{C} \frac{K^7}{\nu^{19}} \left( \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \right)^5 (t^5 + 1), \quad (1.12)$$

where  $\bar{C}$  is independent of  $\nu$ ,  $K$ , and the initial data.

The constants above all depend on  $\inf \theta_0$  in nontrivial ways. The condition on the relative sizes of  $\kappa$  and  $\nu$  is slightly weaker; we only need  $K \geq 2$ . This condition emerges in an interesting way (that is also present in the proof of Theorem 1.1). When we try to do the weighted enstrophy estimate, the diffusion itself produces terms which are roughly like

$$\int \theta^{-\frac{K+1}{K}} |\nabla u|^2 |\nabla \theta|^2 dx.$$

In fact, one such term (named  $J_1$  in both proofs below) has a definite (and detrimental) sign in the resulting inequality. So special cancellations should not be expected.

We also remark that the actual lower bound for  $K$  implied by our methods is a bit lower than 2, but then the dependencies in the constants become more complicated. Theorem 1.2 would remain true with

$$K > \frac{3 + \sqrt{17}}{4} \approx 1.78,$$

and condition (1.5) could be weakened to

$$\kappa + \nu > \max(L\sqrt{\nu}, \frac{3 + \sqrt{17}}{4}\nu).$$

Basically, we just need the term labeled  $U_2$  in inequalities (2.5) and (3.9) below to have a positive coefficient. A slightly different approach (see Remark 2) improves this lower bound further, but it seems that the large Prandtl number regime ( $K \approx 1$ ) will require some fundamentally different approaches.

Although the proof of Theorem 1.2 is similar to the proof for Theorem 1.1, there are two aspects that we believe make it worth presenting. Firstly, the initial thermodynamic a priori estimates are harder to prove and less clear-cut than for the thermal Navier-Stokes system; compare Proposition 2.1 with Proposition 3.1. Compressibility allows for a “refrigeration” effect, where an expanding gas becomes colder. This is seen explicitly in (3.5). Nevertheless, we are still able to prove a (time-decaying) lower bound on the temperature and an upper bound on the total heat  $\|\theta\|_{L^2}^2$  for the thermal reduced Burgers equation (both of which are crucial). Secondly, we were rather surprised that compressibility did not hinder the main argument, since it is fundamentally an energy-type estimate. Beyond Proposition 3.1, the rest of the proof of Theorem 1.2 is no harder than for the incompressible case; bypassing the need for commutator estimates with Riesz operators.

Bootstrapping the above a priori bounds to higher regularity can be done through classical techniques [4]. This is expected, since the new conserved quantities easily satisfy the Ladyzhenskaya-Prodi-Serrin condition ([17], [12]), ignoring for the moment that the dissipation has potentially rough coefficients in the form of  $\theta$ . A more modern (or direct) approach would be along the lines of [16], where Hölder continuity is propagated through the use of Campanato's characterization of Hölder spaces [5]: for  $\varphi \in C_c^\infty(\mathbb{R}^n)$  non-negative, radially symmetric, supported in  $B_1(0)$ , with  $\int \varphi(y)dy = 1$  and  $f \in L^2(\mathbb{R}^n)$ , if there are  $A > 0$  and  $\alpha \in (0, 1)$  such that for all  $r > 0$  and  $x \in \mathbb{R}^n$ ,

$$\int_{B_1(0)} \left| f(x + ry) - \int_{B_1(0)} f(x + rz) \varphi(z) dz \right|^2 \varphi(y) dy \leq A^2 r^{2\alpha},$$

then  $f$  has a Hölder continuous representative in  $C^\alpha(\mathbb{R}^n)$  (and  $[f]_{C^\alpha} \lesssim A$ ). The conserved quantities of (1.7) and (1.11) should suffice to repeat the proof of [16] for  $u$ . Note that, since

$$\int \theta^{(K-1)/K} |\nabla^2 u|^2 dx$$

is time-integrable, then it is in particular finite for a sequence of times that converge to 0. By Sobolev embedding ( $H^2(\mathbb{T}^3) \subset C^{1/2}(\mathbb{T}^3)$ ), the propagation argument of [16] would imply our solutions  $u$  become Hölder continuous instantly, even with  $u_0 \in H^1$ .

Although we only present a strong a priori bound for classical solutions, the global well-posedness of the Cauchy problems (1.1)-(1.2) and (1.8)-(1.9) in Sobolev space should follow from established techniques. The existence of weak solutions could be proven in a similar manner as for the Navier-Stokes-Fourier system (see [7] and [10]). From there, one could prove a local well-posedness result (as in [14] for the Navier-Stokes-Fourier system) to obtain classical solutions on a short time interval that depends sensitively on the initial data. Theorems 1.1 and 1.2 would then extend this solution to a global one. Alternatively, one could mollify the weak solutions and use the results of this paper to show a convergence of mollified solutions to a unique strong solution. The complete proof of global well-posedness is an interesting topic and will be the subject of a forthcoming paper.

## 2. THERMAL NSE

We first prove some essential thermodynamic properties of the system (1.1) and (1.2).

**Proposition 2.1** (Initial bounds on energy and minimal temperature). *For  $u$  and  $\theta$  smooth solutions to (1.1) and (1.2) on the time interval  $[0, T]$ , with initial data  $u_0$  and  $\theta_0 \geq 1$  respectively, we have*

$$\inf_{\mathbb{T}^3} \theta(\cdot, t) \geq 1,$$

for all  $t \in [0, T]$  and

$$\sup_{t \in [0, T]} \left( \frac{1}{2} \|u(\cdot, t)\|_{L^2(\mathbb{T}^3)}^2 + \|\theta(\cdot, t)\|_{L^2(\mathbb{T}^3)}^2 \right) \leq \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^3)}^2 + \|\theta_0\|_{L^2(\mathbb{T}^3)}^2.$$

**PROOF.** The minimum principle for  $\theta$  is intuitive, but we will demonstrate it rigorously. Take  $\widehat{T} > 0$  and  $\epsilon > 0$  to be determined later. Define  $v(x, t) = \theta(x, t) + \epsilon t$ . Then, by (1.3),

$$\partial_t v + u \cdot \nabla v - \kappa \theta \Delta v - 2\kappa |\nabla v|^2 = \frac{\nu}{2} |\nabla u|^2 + \epsilon. \quad (2.1)$$

Since  $v$  is a smooth function and  $\mathbb{T}^3 \times [0, \widehat{T}]$  is compact, it achieves a global minimum; say this happens at  $(\widehat{x}, \widehat{t})$ . If  $\widehat{t} > 0$ , then we know that

$$\partial_t v(\widehat{x}, \widehat{t}) \leq 0, \quad \Delta v(\widehat{x}, \widehat{t}) \geq 0, \quad \text{and} \quad \nabla v(\widehat{x}, \widehat{t}) = 0.$$

However, so long as  $\theta(x, t) \geq 0$ , (2.1) shows that

$$\partial_t v(\hat{x}, \hat{t}) \geq \epsilon .$$

Let  $\tilde{T}$  be the maximum time for which  $\theta \geq 0$  on  $\mathbb{T}^3 \times [0, \tilde{T}]$ . This is positive by continuity and the assumption that  $\theta_0(x) \geq 1$ . Note that this is also a time interval on which (1.3) is valid. Choose  $\hat{T} = \min(\tilde{T}, T)$ . Then the minimum of  $v$  on  $\mathbb{T}^3 \times [0, \hat{T}]$  must happen at time  $\hat{t} = 0$ . But, since  $\epsilon$  was arbitrary, this means

$$\inf_{\mathbb{T}^3 \times [0, \hat{T}]} \theta(x, t) = \inf_{\mathbb{T}^3} \theta_0(x) \geq 1 .$$

In particular, this shows that  $\tilde{T} > \hat{T}$  (again by continuity) and that  $\hat{T} = T$ , proving the first part of Proposition 2.1 (and that (1.3) is an effective substitute for (1.2)).

Integrating (1.1) against  $u_j$  and integrating (1.2) by itself yields

$$\frac{\partial_t}{2} \|u\|_{L^2}^2 + \nu \int \theta |\nabla u|^2 dx = 0 \quad \text{and} \quad \partial_t \|\theta\|_{L^2}^2 = \nu \int \theta |\nabla u|^2 dx .$$

Adding the two shows that

$$\partial_t \left( \frac{1}{2} \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) = 0 ,$$

which completes the proof of the Proposition. •

It is also true that we get an a priori control on the “thermal enstrophy”

$$\nu \int_0^T \int \theta |\nabla u|^2 dx dt \leq \frac{1}{2} \|u_0\|_{L^2}^2 ,$$

as with the regular Navier-Stokes equations. Interestingly, the proof of the main theorem will not require such a bound, though it establishes much stronger conserved quantities.

**PROOF OF THEOREM 1.1.** Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  (to be specified later) be our thermal weight function. We proceed by integrating (1.1) against  $-\operatorname{div}(f(\theta)\nabla u)$  to get:

$$\underbrace{\int f \frac{\partial_t}{2} |\nabla u|^2 dx}_{I_0} - \underbrace{\int u_i \partial_i u_j \partial_k (f \partial_k u_j) dx}_{I_A} - \underbrace{\int \partial_j p \partial_k (f \partial_k u_j) dx}_{I_P} + \underbrace{\nu \int \partial_i (\theta \partial_i u_j) \partial_k (f \partial_k u_j) dx}_{I_D} = 0$$

The advection term is straightforward:

$$I_A = \int \partial_k u_i \partial_i u_j f \partial_k u_j dx + \int u_i \partial_i \partial_k u_j f \partial_k u_j dx \geq - \int f |\nabla u|^3 dx - \frac{1}{2} \int f' |\nabla u|^2 u_i \partial_i \theta dx \quad (2.2)$$

Recalling (1.4), we integrate by parts twice in  $I_P$  to get the formula

$$I_P = - \int R_l R_k [u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m] f' \partial_j \theta \partial_k u_j dx , \quad (2.3)$$

where  $R_j$  is the Riesz operator  $\partial_j (-\Delta)^{-1/2}$  on the torus, with Fourier symbol  $i \frac{k_j}{|k|}$ .

Lastly,  $I_D$  will provide three positive terms (and one cancellation) for our estimates.

$$\begin{aligned}
I_D &= \nu \int \partial_k(\theta \partial_i u_j) \partial_i(f \partial_k u_j) dx = \nu \int (\partial_k \theta \partial_i u_j + \theta \partial_k \partial_i u_j)(f' \partial_i \theta \partial_k u_j + f \partial_i \partial_k u_j) dx \\
&= \nu \int (f' |\nabla \theta \cdot \nabla u|^2 + f \partial_k \theta \partial_i u_j \partial_i \partial_k u_j + \theta f' \partial_i \theta \partial_k u_j \partial_k \partial_i u_j + \theta f |\nabla^2 u|^2) dx \\
&= \underbrace{\nu \int f' |\nabla \theta \cdot \nabla u|^2 dx}_{J_1} + \underbrace{\nu \int (f + \theta f') \partial_k \theta \partial_i u_j \partial_i \partial_k u_j dx}_{J_2} + \underbrace{\nu \int \theta f |\nabla^2 u|^2 dx}_{J_3}
\end{aligned}$$

The last equality was obtained by re-indexing. Observe that  $J_3$  is a positive term. Continuing, we integrate  $J_2$  by parts to move the  $k$ -derivative away from  $\partial_i u_j$  and onto  $\partial_k \theta$ :

$$\begin{aligned}
J_2 &= -\frac{\nu}{2} \int (f + \theta f') \partial_k \partial_k \theta |\nabla u|^2 dx - \frac{\nu}{2} \int \partial_k (f + \theta f') \partial_k \theta |\nabla u|^2 dx \\
&= \frac{\nu}{2\kappa} \int \frac{f + \theta f'}{\theta} (-\kappa \theta \Delta \theta) |\nabla u|^2 dx - \underbrace{\frac{\nu}{2} \int (2f' + \theta f'') |\nabla \theta|^2 |\nabla u|^2 dx}_{K_0} \\
&= \frac{1}{2} \int \left( \frac{\nu}{\kappa} \frac{f + \theta f'}{\theta} \right) \left( \underbrace{\frac{\nu}{2} |\nabla u|^2}_{K_1} + \underbrace{2\kappa |\nabla \theta|^2}_{K_2} - \underbrace{u \cdot \nabla \theta}_{K_3} - \underbrace{\partial_t \theta}_{K_4} \right) |\nabla u|^2 dx - K_0
\end{aligned}$$

The last equality comes from direct substitution with (1.3). We would like to merge the integral involving the time derivative of  $\theta$  (that is, the one coming from the  $K_4$  term) with  $I_0$  to pull the time derivative entirely outside the integral (and actually begin to get a control on a norm). In order to do this, we must clearly have

$$f'(\theta) = -\frac{\nu}{\kappa} \frac{f(\theta) + \theta f'(\theta)}{\theta} \quad \text{or} \quad f(\theta) = K \theta^{-\frac{1}{K}} \quad \text{and} \quad f'(\theta) = -\theta^{-\frac{K+1}{K}}.$$

**Remark 2:** The more conventional method for computing energy estimates with weights would dictate that we start with  $\int f(\theta) |\nabla u|^2 dx$ , take a time derivative, then use equations (1.1) and (1.3) to derive the inequality. This approach does not, at first, impose any condition on the weight  $f$ . The method above is basically the same, but we demanded that the term arising from  $-\kappa \theta \Delta \theta$  in (1.3) exactly cancels with a term (containing  $\Delta \theta$ ) that arises from integrating the dissipation of (1.1) by parts; this fixes  $f$  as above. We mention that, using the conventional approach outlined in this remark, we could not produce a better final inequality or even a better class of admissible weights.

Having established  $f(\theta)$  explicitly, we now make some observations. As mentioned before,

$$I_0 + \frac{1}{2} \int \left( \frac{\nu}{\kappa} \frac{f + \theta f'}{\theta} \right) K_4 |\nabla u|^2 dx = \frac{\partial_t}{2} \int K \theta^{-\frac{1}{K}} |\nabla u|^2 dx.$$

Moreover, the second term in (2.2) from  $I_A$  exactly cancels with the integral arising from  $K_3$ .

Also,  $K_0$  and the integral arising from  $K_2$  combine into

$$\nu \left( K - \frac{1}{2} - \frac{1}{2K} \right) \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx \quad (2.4)$$

which is also positive.

$J_1$  is evidently negative. Observe, however, that it is controlled by (2.4) (since  $\sum_{i,j} a_i b_i a_j b_j \leq \sum_{i,j} a_i^2 b_j^2$ ), provided  $K - 1/(2K) > 3/2$ ; which certainly holds if  $K \geq 2$ . This is the first time we see the methodology's dependence on  $K$ .



The crucial benefit from this substitution is seen in the integral term arising from  $K_1$ . This gives us a gradient of  $u$  to the *fourth* power with a positive sign (weighed by a negative power of  $\theta$  that improves as  $K$  increases). Combined with (2.4), these two positive terms allow us to close the estimate in a novel way.

**Remark 3:** We mention here that the above method requires that the viscosity and conductivity both grow with temperature. If, in place of (1.1) and (1.2), we consider the system

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p - \nu \operatorname{div}(w(\theta) \nabla u) &= 0 \\ \partial_t(\theta^2) + u \cdot \nabla(\theta^2) - \kappa \operatorname{div}(w(\theta) \nabla(\theta^2)) &= \nu w(\theta) |\nabla u|^2\end{aligned}$$

for  $w$  a smooth positive weight, then we could proceed in a similar fashion up until the analysis of the term  $J_2$ . In order for the cancellation to occur properly, we would need

$$\frac{\nu}{\kappa} \frac{f w' + f' w}{w} = -f',$$

so that (up to a multiplicative constant)

$$f(\theta) = K w(\theta)^{-1/K}.$$

If  $f'$  were not strictly negative, we would no longer have a gain from the  $K_1$  and  $K_2$  terms; indeed, those terms would have a bad sign if  $f$  were increasing in  $\theta$ . Since

$$f'(\theta) = -w(\theta)^{-(K+1)/K} w'(\theta),$$

we see that the above improvement can only happen if  $w' > 0$ .

Collecting what terms remain, we obtain the inequality:

$$\begin{aligned}\frac{\partial_t}{2} \int K \theta^{-\frac{1}{K}} |\nabla u|^2 dx + \underbrace{\nu K \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx}_{U_1} + \underbrace{\nu \frac{2K^2 - 3K - 1}{2K} \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx}_{U_2} \\ + \underbrace{\frac{\nu}{4} \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx}_{U_3} \leq K \int \theta^{-\frac{1}{K}} |\nabla u|^3 dx - I_P\end{aligned}\quad (2.5)$$

We use Hölder's inequality and Sobolev embedding in three dimensions to control the cubic term on the right hand side.

$$\begin{aligned}K \int \theta^{-\frac{1}{K}} |\nabla u|^3 dx &\leq K \left( \int |\nabla u|^6 dx \right)^{\frac{1}{9}} \left( \int |\nabla u|^4 \theta^{-\frac{K+1}{K}} dx \right)^{\frac{7}{12}} \left( \int \theta^2 dx \right)^{\frac{7K-5}{24K}} |\mathbb{T}^3|^{\frac{7K-3}{24K}} \\ &\leq CK \left( \int |\nabla^2 u|^2 dx \right)^{\frac{1}{3}} \left( \int |\nabla u|^4 \theta^{-\frac{K+1}{K}} dx \right)^{\frac{7}{12}} \left( \int \theta^2 dx \right)^{\frac{7K-5}{24K}}\end{aligned}$$

where  $C$  incorporates the Sobolev constant and the size of the torus; taking  $C$  large enough also makes it independent of  $K$ . Using the minimum principle for  $\theta$  and Young's inequality with exponents 3,  $\frac{12}{7}$ , and 12, we see that

$$K \int \theta^{-\frac{1}{K}} |\nabla u|^3 dx \leq \frac{1}{3} U_1 + \frac{1}{3} U_3 + C \frac{K^8}{\nu^{11}} \|\theta\|_{L^2}^{\frac{7K-5}{K}} \quad (2.6)$$

for a different constant  $C$  independent of  $K$  and  $\nu$ . Also recall that

$$\|\theta\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2 + \frac{1}{2} \|u_0\|_{L^2}^2.$$

Thus, the cubic term on the right hand side of (2.5) is indeed dominated by the positive terms and the conserved energy.



We now quote a classical commutator estimate by [6]: for  $R_k$  a Riesz operator and  $\varphi, g \in C_0^\infty(\mathbb{R}^n)$ , if

$$[R_k, \varphi]g := R_k(\varphi g) - \varphi R_k g ,$$

then

$$\|[R_k, \varphi]g\|_{L^p} \lesssim [\varphi]_{BMO} \|g\|_{L^p} ,$$

where the constant implied by the inequality only depends on  $p$  and the dimension  $n$ . Theorem I of [6] covers more general Calderón-Zygmund singular integral operators (such as the composite  $R_k R_l$ ), but for our purposes we quote the above bound and observe that

$$[R_k R_l, \varphi](g) = R_k ([R_l, \varphi](g)) + [R_k, \varphi] (R_l g) .$$

By boundedness of Riesz transforms on  $L^p$  for  $1 < p < \infty$  (see for instance [18]), we see that

$$\|[R_k R_l, \varphi](g)\|_{L^2(\mathbb{T}^3)} \leq c[\varphi]_{BMO} \|g\|_{L^2(\mathbb{T}^3)} , \quad (2.7)$$

with  $c$  independent of  $\varphi$  or  $g$ .

Now we estimate  $I_P$ . From (2.3) and the explicit form of  $f$ , we have that

$$\begin{aligned} I_P &= \int R_k R_l (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \theta^{-\frac{K+1}{2K}} \partial_j \theta \partial_k u_j dx \\ &= \int R_k R_l \left( \theta^{-\frac{K+1}{2K}} (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \right) \theta^{-\frac{K+1}{2K}} \partial_j \theta \partial_k u_j dx \\ &\quad - \int \left[ R_k R_l, \theta^{-\frac{K+1}{2K}} \right] (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \theta^{-\frac{K+1}{2K}} \partial_j \theta \partial_k u_j dx \end{aligned}$$

By Young's inequality and the boundedness of Riesz operators,

$$\begin{aligned} &\left| \int R_k R_l \left( \theta^{-\frac{K+1}{2K}} (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \right) \theta^{-\frac{K+1}{2K}} \partial_j \theta \partial_k u_j dx \right| \\ &\leq \frac{1}{2\nu} \left\| R_k R_l \left( \theta^{-\frac{K+1}{2K}} (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \right) \right\|_{L^2}^2 + \frac{\nu}{2} \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx \\ &\leq \frac{C}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx + C\nu \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx . \end{aligned} \quad (2.8)$$

Since  $\theta$  is bounded below uniformly,  $\theta^{-(K+1)/(2K)}$  is an  $L^\infty$  function. For any smooth function,  $[f]_{BMO} \leq 2\|f\|_{L^\infty}$ , so

$$\left[ \theta^{-\frac{K+1}{2K}} \right]_{BMO} \leq 2(\inf \theta)^{-\frac{K+1}{2K}} \leq 2 ,$$

which we incorporate into the ambient constants. By (2.7), we then have

$$\begin{aligned} &\left| \int \left[ R_k R_l, \theta^{-\frac{K+1}{2K}} \right] (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \theta^{-\frac{K+1}{2K}} \partial_j \theta \partial_k u_j dx \right| \\ &\leq \frac{1}{2\nu} \left\| \left[ R_k R_l, \theta^{-\frac{K+1}{2K}} \right] \left( \theta^{-\frac{K+1}{2K}} (u_m \partial_m u_l - \nu \partial_m \theta \partial_l u_m) \right) \right\|_{L^2}^2 + \frac{\nu}{2} \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx \\ &\leq \frac{C}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx + C\nu \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx . \end{aligned} \quad (2.9)$$

Combining (2.8) and (2.9), we see that

$$|I_P| \leq \frac{C}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx + \frac{2}{3} U_2 , \quad (2.10)$$

assuming (1.5). It is here that we essentially determine the constant in (1.5). Note that it only depends on bounds for Riesz operators and their commutators (and, technically, on  $\inf \theta$ ).

To complete the bound, we estimate

$$\int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx \leq \left( \int |u|^4 dx \right)^{1/2} \left( \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx \right)^{1/2} \|\theta^{-\frac{K+1}{2K}}\|_{L^\infty}.$$

Agmon's inequality (Lemma 13.2 of [1]) on a compact three-dimensional domain says that

$$\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{1/4} \|u\|_{H^2}^{3/4}.$$

Thus, by Hölder's inequality and monotonicity of kinetic energy,

$$\|u\|_{L^4}^4 \leq \|u\|_{L^2}^2 \|u\|_{L^\infty}^2 \leq C \|u_0\|_{L^2}^{5/2} \left( \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + \|u_0\|_{L^2}^2 \right)^{3/4},$$

for yet another constant  $C$  independent of all parameters. Therefore

$$\begin{aligned} \frac{C}{\nu} \int \theta^{-\frac{K+1}{K}} |u|^2 |\nabla u|^2 dx &\leq \frac{\nu}{12} \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx + \frac{C}{\nu^3} \|u_0\|_{L^2}^{5/2} \left( \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + \|u_0\|_{L^2}^2 \right)^{3/4} \\ &\leq \frac{1}{3} U_3 + \frac{1}{3} U_1 + K\nu \|u_0\|_{L^2}^2 + \frac{C}{K^3 \nu^{15}} \|u_0\|_{L^2}^{10}, \end{aligned} \quad (2.11)$$

where  $C$  has become larger but is still independent of  $K$ ,  $\nu$ , or the initial data.

Combining all three estimates ((2.6), (2.10), and (2.11)), inequality (2.5) becomes

$$\frac{\partial_t}{2} \int K \theta^{-\frac{1}{K}} |\nabla u|^2 dx + \frac{1}{3} (U_1 + U_2 + U_3) \leq \nu K \|u_0\|_{L^2}^2 + C \frac{K^8}{\nu^{11}} (\|\theta_0\|_{L^2}^2 + \frac{1}{2} \|u_0\|_{L^2}^2)^{\frac{7K-5}{2K}} + \frac{C}{K^3 \nu^{15}} \|u_0\|_{L^2}^{10}.$$

Integrating this inequality yields (1.6) and (1.7). •

### 3. THERMAL REDUCED BURGERS

We begin by proving some thermodynamic estimates that are similar to Proposition 2.1. The lack of incompressibility, however, will make the a priori bounds more complicated and will also reduce the strength of the minimum principle for the temperature.

**Proposition 3.1** (Initial bounds on velocity and temperature). *Let  $\gamma \geq 2$ . For  $u$  and  $\theta$  smooth solutions to (1.8) and (1.9) on the time interval  $[0, T]$ , with initial data  $u_0$  and  $\theta_0 \geq 1$  respectively, we have*

$$\theta(x, t) \geq \frac{1}{3t/(8\nu) + 1}, \quad (3.1)$$

for all  $(x, t) \in \mathbb{T}^3 \times [0, T]$  and

$$\begin{aligned} \sup_{t \in [0, T]} \frac{1}{\gamma} \|u(\cdot, t)\|_{L^\gamma(\mathbb{T}^3)}^\gamma + \nu \int_0^T \int_{\mathbb{T}^3} \theta(x, t) |\nabla u(x, t)|^2 |u(x, t)|^{\gamma-2} dx dt \\ + \frac{4\nu(\gamma-2)}{\gamma^2} \int_0^T \int_{\mathbb{T}^3} \theta(x, t) |\nabla |u(x, t)|^{\gamma/2}|^2 dx dt \leq \frac{1}{\gamma} \|u_0\|_{L^\gamma(\mathbb{T}^3)}^\gamma. \end{aligned} \quad (3.2)$$

Moreover, if  $9/4 \leq \gamma \leq 6$  or  $\gamma = 2$ , we also have

$$\sup_{t \in [0, T]} \left( \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 + \|\theta(\cdot, t)\|_{L^2}^2 \right) \leq \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 + C(T^{3/2} + 1) \left( \|u_0\|_{L^\gamma}^{3\gamma-3} + 1 \right). \quad (3.3)$$

PROOF. We begin with (3.1), providing a “minimum local strength” for the thermal viscosity. As with the incompressible case, it will be convenient from now on to work with  $\theta$  instead of the actual temperature; (1.9) then reduces to

$$\partial_t \theta + \frac{1}{2} \operatorname{div}(u) \theta + u_i \partial_i \theta - \kappa \theta \Delta \theta = 2\kappa |\nabla \theta|^2 + \frac{\nu}{2} |\nabla u|^2, \quad (3.4)$$

on any time interval where  $\theta \geq 0$ . Let the maximum such time interval be  $[0, \tilde{T}]$ . As in the proof of Proposition 2.1,  $\tilde{T} > 0$  by continuity and the assumption that  $\theta_0 \geq 1$ . Define  $\hat{T} = \min(T, \tilde{T})$  and define

$$v(x, t) = \left( \frac{3t}{8\nu} + 1 \right) \theta(x, t).$$

Then, on  $\mathbb{T}^3 \times [0, \hat{T}]$ ,  $v$  satisfies the equation

$$\partial_t v + u \cdot \nabla v - \kappa \theta \Delta v - \frac{2\kappa}{3t/(8\nu) + 1} |\nabla v|^2 = \frac{1}{2} \left( \frac{3t}{8\nu} + 1 \right) (\nu |\nabla u|^2 - \operatorname{div}(u) \theta) + \frac{3}{8\nu} \theta. \quad (3.5)$$

Since  $v$  is smooth and  $\mathbb{T}^3 \times [0, \hat{T}]$  is compact,  $v$  achieves its global minimum; say at  $(\hat{x}, \hat{t})$ . If  $\hat{t} > 0$ , then

$$\partial_t v(\hat{x}, \hat{t}) \leq 0, \quad \Delta v(\hat{x}, \hat{t}) \geq 0, \quad \text{and} \quad \nabla v(\hat{x}, \hat{t}) = 0.$$

By Young's inequality,

$$|\operatorname{div}(u) \theta| \leq \frac{\nu}{3} (\operatorname{div}(u))^2 + \frac{3}{4\nu} \theta^2 \leq \nu |\nabla u|^2 + \frac{3}{4\nu} \theta^2.$$

We then have from (3.5) that

$$\partial_t v(\hat{x}, \hat{t}) \geq \left( \frac{3t}{8\nu} + 1 \right) \left( -\frac{3\theta^2}{8\nu} \right) + \frac{3\theta}{8\nu} = \frac{v(\hat{x}, \hat{t}) - v(\hat{x}, \hat{t})^2}{t + 8\nu/3}.$$

Assume that  $v(\hat{x}, \hat{t}) < 1$ . Then  $\partial_t v(\hat{x}, \hat{t})$  is strictly positive, and this can only happen if  $\hat{t} = 0$ . But then

$$\inf_{\mathbb{T}^3 \times [0, \hat{T}]} v(x, t) = \inf_{\mathbb{T}^3} v(x, 0) = \inf_{\mathbb{T}^3} \theta_0(x) \geq 1,$$

which is a contradiction. Therefore  $v(x, t) \geq 1$  on  $\mathbb{T}^3 \times [0, \hat{T}]$ . But this shows that  $\tilde{T} > \hat{T}$ , so that  $\hat{T} = T$ , and that (3.1) holds.

To prove (3.2), we multiply (1.8) by  $u_j |u|^{\gamma-2}$ , sum in  $j$ , and integrate in space. Integration by parts shows that the nonlinearities (in  $u$ ) cancel each other, leaving

$$\frac{\partial_t}{\gamma} \|u\|_{L^\gamma}^\gamma + \nu \int \theta |\nabla u|^2 |u|^{\gamma-2} dx + \frac{4\nu(\gamma-2)}{\gamma^2} \int \theta |\nabla |u|^{\gamma/2}|^2 dx = 0.$$

Recall that  $\theta$  stays positive. Integrating this in time from 0 to  $\bar{T}$ , then taking a supremum in  $\bar{T}$  over  $[0, T]$  yields (3.2).

Lastly, our main estimates will require some manner of upper bound on  $\theta$ . It will suffice to have a bound on the “total heat” in the system, given by  $\|\theta\|_{L^2}$ . To do this, we multiply (1.8) by  $u_j$ , sum in  $j$ , and integrate in space (as if  $\gamma$  were 2). This gives

$$\frac{\partial_t}{2} \|u\|_{L^2}^2 + \nu \int \theta |\nabla u|^2 dx = \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int |u|^2 \operatorname{div}(u) dx.$$

If we integrate (1.9) in space and add it to the above, we get

$$\partial_t \left( \frac{1}{2} \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) = \frac{\gamma-2}{2\gamma} \int |u|^2 \operatorname{div}(u) dx \leq \frac{1}{2} (3t/(8\nu) + 1)^{1/2} \int \theta^{1/2} |\nabla u| |u|^2 dx.$$

Note that, if  $\gamma = 2$ , we don't need the last inequality. We actually have true conservation of kinetic and thermal energy, and (3.3) takes on the simpler form

$$\frac{1}{2}\|u(\cdot, t)\|_{L^2}^2 + \|\theta(\cdot, t)\|_{L^2}^2 = \frac{1}{2}\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2.$$

For the rest of the proof, we assume  $\gamma \geq 9/4$ . Since  $\gamma \leq 6$ , we use Hölder's inequality to get

$$\frac{1}{2}(3t/(8\nu) + 1)^{1/2} \int \theta^{1/2} |\nabla u| |u|^2 dx \leq \left( \frac{3t + 8\nu}{32\nu} \right)^{1/2} \left( \int \theta |\nabla u|^2 |u|^{\gamma-2} dx + \int |u|^{6-\gamma} dx \right).$$

We see that the first term is time-integrable by (3.2). If  $\gamma \geq 3$ , we have that

$$\int_{\mathbb{T}^3} |u|^{6-\gamma} dx \leq C \|u\|_{L^\gamma}^\gamma,$$

for some fixed  $C > 0$ . Integrating on  $[0, T]$  and taking a supremum in time then yields

$$\sup_{t \in [0, T]} \left( \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 + \|\theta(\cdot, t)\|_{L^2}^2 \right) \leq \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 + C(T + 1) \|u_0\|_{L^\gamma}^\gamma, \quad (3.6)$$

for a different fixed constant  $C > 0$  that depends on  $\nu$  and  $\gamma$ . We can improve the range of  $\gamma$  by employing the third term of (3.2). Recalling the Sobolev embedding of  $H^2(\mathbb{T}^3)$  into  $L^6(\mathbb{T}^3)$ , there is some constant  $C > 0$  such that

$$\|u\|_{L^{3\gamma}}^\gamma = \left( \int |u|^{3\gamma} dx \right)^{1/3} \leq C \int |\nabla |u|^{\gamma/2}|^2 dx + C \|u\|_{L^\gamma}^\gamma \leq C \frac{3t + 8\nu}{8\nu} \int \theta |\nabla |u|^{\gamma/2}|^2 dx + C \|u\|_{L^\gamma}^\gamma.$$

For  $3 > \gamma > 3/2$ , we have that  $\gamma < 6 - \gamma < 3\gamma$ . Thus we may interpolate

$$\|u\|_{L^{6-\gamma}} \leq \|u\|_{L^{3\gamma}}^\alpha \|u\|_{L^\gamma}^{1-\alpha},$$

where  $\alpha = (9 - 3\gamma)/(6 - \gamma)$ . Finally, since we assume  $\gamma \geq 9/4$ , we have that  $\alpha(6 - \gamma) = 9 - 3\gamma \leq \gamma$ . Therefore,

$$\begin{aligned} \int |u|^{6-\gamma} dx &\leq C \|u\|_{L^\gamma}^{2\gamma-3} \left( \frac{3t + 8\nu}{8\nu} \int \theta |\nabla |u|^{\gamma/2}|^2 dx + \|u\|_{L^\gamma}^\gamma \right)^{9/\gamma-3} \\ &\leq C \|u\|_{L^\gamma}^{2\gamma-3} \left( \frac{3t + 8\nu}{8\nu} \int \theta |\nabla |u|^{\gamma/2}|^2 dx + \|u\|_{L^\gamma}^\gamma + 1 \right) \end{aligned}$$

Integrating on  $[0, T]$  and taking a supremum in time then yields

$$\sup_{t \in [0, T]} \left( \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 + \|\theta(\cdot, t)\|_{L^2}^2 \right) \leq \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 + C(T^{3/2} + 1) \|u_0\|_{L^\gamma}^{3\gamma-3}.$$

This, together with (3.6) yields (3.3). •

The proof of the weighted enstrophy bound for the thermal reduced Burgers equation will be very similar to that for the thermal Navier-Stokes equations. Nevertheless, we believe it is worth investigating since it is interesting to see how the lack of a divergence-free condition on  $u$  does not prevent the inequality from closing, despite being an energy-type argument.

**PROOF OF THEOREM 1.2:** Although the statement of Theorem 1.2 provides the thermal weight explicitly, we will again perform the calculations with a generic weight  $f(\theta)$ . There will come a point where  $f$  will have to satisfy a specific ODE in order for the estimate to proceed; at that point we will restrict to the weight mentioned in the theorem. We do this to emphasize how (and exactly where) the methodology determines  $f$ .

We start by integrating (1.8) against  $-\operatorname{div}(f\nabla u)$  to get:

$$\underbrace{\int f \frac{\partial_t}{2} |\nabla u|^2 dx}_{I_0} - \underbrace{\int u_i \partial_i u_j \partial_k (f \partial_k u_j) dx - \frac{1}{\gamma} \int \operatorname{div}(u) u_j \partial_k (f \partial_k u_j) dx}_{I_A} + \underbrace{\nu \int \partial_i (\theta \partial_i u_j) \partial_k (f \partial_k u_j) dx}_{I_D} = 0$$

The advection terms are straightforward:

$$\begin{aligned} I_A &= \int f \partial_k u_i \partial_i u_j \partial_k u_j dx + \int f u_i \partial_i \left( \frac{|\nabla u|^2}{2} \right) dx + \frac{1}{\gamma} \int f \operatorname{div}(u) |\nabla u|^2 dx + \frac{1}{\gamma} \int f \partial_k \operatorname{div}(u) \partial_k u_j u_j dx \\ &\leq -\frac{3\gamma-2}{2\gamma} \int f |\nabla u|^3 dx - \frac{1}{2} \int u_i \partial_i f |\nabla u|^2 dx - \frac{1}{\gamma} \int f |\nabla^2 u| |\nabla u| |u| dx \end{aligned} \quad (3.7)$$

Now we examine  $I_D$ , which will provide three positive terms (and one cancellation) for our estimates.

$$\begin{aligned} I_D &= \nu \int \partial_k (\theta \partial_i u_j) \partial_i (f \partial_k u_j) dx = \nu \int (\partial_k \theta \partial_i u_j + \theta \partial_k \partial_i u_j) (f' \partial_i \theta \partial_k u_j + f \partial_i \partial_k u_j) dx \\ &= \nu \int (f' |\nabla \theta \cdot \nabla u|^2 + f \partial_k \theta \partial_i u_j \partial_i \partial_k u_j + \theta f' \partial_i \theta \partial_k u_j \partial_k \partial_i u_j + \theta f |\nabla^2 u|^2) dx \\ &= \underbrace{\nu \int f' |\nabla \theta \cdot \nabla u|^2 dx}_{J_1} + \underbrace{\nu \int (f + \theta f') \partial_k \theta \partial_i u_j \partial_i \partial_k u_j dx}_{J_2} + \underbrace{\nu \int \theta f |\nabla^2 u|^2 dx}_{J_3} \end{aligned}$$

The last equality was obtained by re-indexing. Observe that  $J_3$  is a positive term and that  $J_1$  has a sign depending on  $f'$ . Continuing, we integrate  $J_2$  by parts to move the  $k$ -derivative away from  $\partial_i u_j$  and onto  $\partial_k \theta$ :

$$\begin{aligned} J_2 &= -\frac{\nu}{2} \int (f + \theta f') \partial_k \partial_k \theta |\nabla u|^2 dx - \frac{\nu}{2} \int \partial_k (f + \theta f') \partial_k \theta |\nabla u|^2 dx \\ &= \frac{\nu}{2\kappa} \int \frac{f + \theta f'}{\theta} (-\kappa \theta \Delta \theta) |\nabla u|^2 dx - \underbrace{\frac{\nu}{2} \int (2f' + \theta f'') |\nabla \theta|^2 |\nabla u|^2 dx}_{K_0} \\ &= \frac{1}{2} \int \left( \frac{\nu}{\kappa} \frac{f + \theta f'}{\theta} \right) \left( \underbrace{\frac{\nu}{2} |\nabla u|^2}_{K_1} + \underbrace{2\kappa |\nabla \theta|^2}_{K_2} - \underbrace{u \cdot \nabla \theta}_{K_3} - \underbrace{\frac{\operatorname{div}(u) \theta}{2}}_{K_4} - \underbrace{\partial_t \theta}_{K_5} \right) |\nabla u|^2 dx - K_0 \end{aligned}$$

The last equality comes from direct substitution with (3.4). We would like to merge the integral involving the time derivative of  $\theta$  (that is, the one coming from the  $K_5$  term) with  $I_0$  to pull the time derivative entirely outside the integral. In order to do this, we must clearly have

$$\frac{\nu}{\kappa} \frac{f(\theta) + \theta f'(\theta)}{\theta} = -f'(\theta) \quad \text{or} \quad f(\theta) = K \theta^{-\frac{1}{K}} \quad \text{and} \quad f'(\theta) = -\theta^{-\frac{K+1}{K}}.$$

Having established  $f(\theta)$  explicitly, we now make some observations. As mentioned before,

$$I_0 + \frac{1}{2} \int \left( \frac{\nu}{\kappa} \frac{f + \theta f'}{\theta} \right) K_5 |\nabla u|^2 dx = \frac{\partial_t}{2} \int K \theta^{-\frac{1}{K}} |\nabla u|^2 dx.$$

Moreover, the second term in (3.7) from  $I_A$  exactly cancels with the integral arising from  $K_3$ .

Also,  $K_0$  and the integral arising from  $K_2$  combine into

$$\nu \left( K - \frac{1}{2} - \frac{1}{2K} \right) \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx \quad (3.8)$$

which is also positive.

$J_1$  is again negative, but also controlled by (3.8) provided  $K - 1/(2K) > 3/2$ .

The term arising from  $K_4$  is comparable to the first term in (3.7) from  $I_A$ . We therefore combine them.

Collecting what terms remain, we obtain the inequality:

$$\begin{aligned} & \frac{\partial_t}{2} \int K \theta^{-\frac{1}{K}} |\nabla u|^2 dx + \underbrace{\nu K \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx}_{U_1} + \underbrace{\nu \frac{2K^2 - 3K - 1}{2K} \int \theta^{-\frac{K+1}{K}} |\nabla \theta|^2 |\nabla u|^2 dx}_{U_2} \\ & + \underbrace{\frac{\nu}{4} \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx}_{U_3} \leq \underbrace{\left( \frac{3}{2}K + \frac{1}{4} \right) \int \theta^{-\frac{1}{K}} |\nabla u|^3 dx}_{L_1} + \underbrace{\frac{K}{2} \int \theta^{-\frac{1}{K}} |u| |\nabla u| |\nabla^2 u| dx}_{L_2} \end{aligned} \quad (3.9)$$

By Hölder's inequality,

$$L_1 \leq 2K \left( \int |\nabla u|^6 dx \right)^{\frac{1}{9}} \left( \int |\nabla u|^4 \theta^{-\frac{K+1}{K}} dx \right)^{\frac{7}{12}} \left( \int \theta^2 dx \right)^{\frac{7K-5}{24K}} |\mathbb{T}^3|^{\frac{7K-3}{24K}}.$$

Define

$$\tau(t, \nu) := (3t/(8\nu) + 1).$$

We then use Young's inequality with exponents  $3$ ,  $\frac{12}{7}$ , and  $12$  to obtain

$$L_1 \leq \frac{\nu K}{4\tau^{\frac{K-1}{K}}} \left( \int |\nabla u|^6 dx \right)^{\frac{1}{3}} + \frac{\nu}{12} \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx + C \frac{K^8 \tau^{\frac{4(K-1)}{K}}}{\nu^{11}} \|\theta\|_{L^2}^{\frac{7K-5}{K}} |\mathbb{T}^3|^{\frac{7K-3}{2K}}$$

for some constant  $C$  independent of  $K$ ,  $\nu$ , and  $\tau$ , and every  $t \in [0, T]$ . By (3.3), we have

$$\int \theta^2 dx \leq \|\theta_0\|_{L^2}^2 + \frac{1}{2} \|u_0\|_{L^2}^2 + C(T^{3/2} + 1) \left( \|u_0\|_{L^\gamma}^{3\gamma-3} + 1 \right),$$

and, by Sobolev embedding in three dimensions and the minimum principle (3.1) for  $\theta$ ,

$$\left( \int |\nabla u|^6 dx \right)^{\frac{1}{3}} \leq \int |\nabla^2 u|^2 dx \leq \tau^{\frac{K-1}{K}} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx,$$

we see that  $L_1$  of (3.9) is dominated by the positive terms ( $\frac{1}{4}U_1$  and  $\frac{1}{3}U_3$ ) and the conserved quantities of Proposition (3.1).

Moving on to the last term, Hölder's inequality shows that

$$L_2 \leq \frac{K}{2} \left( \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx \right)^{\frac{1}{2}} \left( \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx \right)^{\frac{1}{4}} \left( \int |u|^4 dx \right)^{\frac{1}{4}} \|\theta^{-\frac{K+1}{4K}}\|_{L^\infty}.$$

Then by Young's inequality and (3.1),

$$L_2 \leq \frac{\nu K}{4} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + \frac{\nu}{12} \int \theta^{-\frac{K+1}{K}} |\nabla u|^4 dx + C \frac{K}{\nu^3} \tau^{\frac{K+1}{4K}} \int |u|^4 dx. \quad (3.10)$$

The first and second terms are bounded by  $\frac{1}{4}U_1$  and  $\frac{1}{3}U_3$ . For the last term, if  $\gamma \geq 4$  we have  $\|u\|_{L^4}^4 \leq C\|u\|_{L^\gamma}^\gamma$ , and we use (3.2). If  $9/4 \leq \gamma < 4$  (or  $\gamma = 2$ ), we use interpolation between  $L^\gamma$  and  $L^\infty$ :

$$\|u\|_{L^4} \leq \|u\|_{L^\gamma}^{\gamma/4} \|u\|_{L^\infty}^{1-\gamma/4}.$$

Agmon's inequality on a compact three-dimensional domain says that

$$\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{1/4} \|u\|_{H^2}^{3/4}.$$

We have from before that

$$\|u\|_{H^2}^2 \leq \tau^{\frac{K-1}{K}} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx,$$

hence

$$\|u\|_{L^4}^4 \leq C \|u\|_{L^\gamma}^\gamma \|u\|_{L^2}^{1-\gamma/4} \left( \tau^{\frac{K-1}{K}} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + \|u\|_{L^2}^2 \right)^{\frac{12-3\gamma}{8}}.$$

Since  $\gamma$  is in all cases larger than  $4/3$ , we have that  $12 - 3\gamma < 8$ . So by Young's inequality again,

$$C \frac{K}{\nu^3} \tau^{\frac{K+1}{4K}} \|u\|_{L^4}^4 \leq \frac{\nu K}{4} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + C \frac{K}{\nu^3} \tau^{\frac{K+1}{4K}} \|u\|_{L^\gamma}^{\frac{8\gamma}{3\gamma-4}} \|u\|_{L^2}^{\frac{8-2\gamma}{3\gamma-4}} \left( \tau^{\frac{5K-3}{4K}} \nu^{-4} \right)^{\frac{12-3\gamma}{3\gamma-4}}, \quad (3.11)$$

where the  $C$  on the right is a different constant from the one on the left. Combining this with (3.10), we see that  $L_2$  of (3.9) is bounded similarly to  $L_1$ .

Inequality (3.9) then reduces to

$$\frac{K}{2} \partial_t \int \theta(x, t)^{-\frac{1}{K}} |\nabla u(x, t)|^2 dx + \frac{1}{4} U_1 + U_2 + \frac{1}{3} U_3 \leq C(\nu, \gamma, K, \|u_0\|_{L^\gamma}, \|\theta_0\|_{L^2}) \left( t^{M(K, \gamma)-1} + 1 \right),$$

for some exponent  $M(K, \gamma)$  which can be computed explicitly. Integrating this ODE immediately gives (1.10) and (1.11).

In the special case  $\gamma = 2$ , (3.11) becomes

$$C \frac{K}{\nu^3} \tau^{\frac{K+1}{4K}} \|u\|_{L^4}^4 \leq \frac{\nu K}{4} \int \theta^{\frac{K-1}{K}} |\nabla^2 u|^2 dx + C \frac{K}{\nu^{15}} \|u\|_{L^2}^{10} \tau^{\frac{4K-2}{K}},$$

and we have strict conservation of the quantity

$$\frac{1}{2} \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2$$

in time. Hence (3.9) effectively becomes

$$\frac{K}{2} \partial_t \int \theta(x, t)^{-\frac{1}{K}} |\nabla u(x, t)|^2 dx + \frac{1}{4} U_1 + U_2 + \frac{1}{3} U_3 \leq \bar{C} \frac{K^8}{\nu^{19}} \left( \frac{1}{2} \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \right)^5 (t^4 + 1),$$

where the above constant  $\bar{C}$  is independent of  $\nu$ ,  $K$ , and the initial data. From this, (1.12) now follows. •

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